Indian Statistical Institute, Bangalore.
Mid-Sem. Supplementary Exam : Graph Theory

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Max. points : 15. **Time Limit : 1.5 hours.**

Answer any three questions only. All questions carry 5 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly. See the end of the question paper for notations.

1. Consider the graph $\mathcal{Q}_n$. Which of the following four inequalities hold and which of them are strict ?

$$3 \leq g(\mathcal{Q}_n) \leq 2diam(\mathcal{Q}_n) + 1 \ ; \ rad(\mathcal{Q}_n) \leq diam(\mathcal{Q}_n) \leq 2rad(\mathcal{Q}_n).$$

2. Let $d = (d_1, \ldots, d_n), n \geq 1$ be a sequence of non-negative integers arranged in non-decreasing order. Show that d is a degree sequence of a forest with exactly k components if and only if $\sum_i d_i = (2n-2k)$. Give a direct proof i.e., without using Cayley's theorem or tree counting theorem or Prufer's code.

3. Consider K_n with integer weights and let the total weight on every triangle be even. Show the subgraph consisting of edges with odd weights forms a spanning complete bipartite subgraph.

4. Show that $2\alpha'(G) = \min_{S \subset V}\{|V(G)| - d(S)\}$ where $d(S) = o(G-S) - |S|$.

Some notations :

- G is assumed to be a finite simple graph everywhere.
- $g(G)$ - Girth of the graph, the length of the smallest cycle if it exists. If no cycle exists, the girth is set to be infinity.

- d_G is defined as the usual graph metric when all edge weights are taken to be 1 and $diam(G) := \max\{d_G(u, v) : u, v \in V(G)\}$.
- For a graph G, $rad(G) \leq r$ if there exists a vertex u such that $V \subset B_r(u)$ i.e., $rad(G) := \min_{u \in V} \max\{d(u, v) : v \in V\}$.
- Q_n is the hypercube graph on $\{0, 1\}^n$ i.e., $V = \{0, 1\}^n$ and $x \sim y$ is x and y differ exactly in one-coordinate i.e., $|\{i : x_i \neq y_i\}| = 1$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
- $\alpha'(G)$ - Maximum independent edge set ; $\beta'(G)$ - Minimum edge cover.
- $\alpha(G)$ - Maximum independent vertex set ; $\beta(G)$ - Minimum vertex cover.
- $o(G)$ - The number of odd-components in a graph.